On shelling and flag vectors

Jonathan Fine*

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Abstract

This note defines a flag vector for i-graphs. The construction applies to any finite combinatorial object that can be shelled. Two possible connections to quantum topology are mentioned. Further details appear in the author's *On quantum topology, hypergraphs and flag vectors*, (preprint q-alg/9708001).

The purpose of this note is to state concisely a new combinatorial concept. This note is an announcement; details appear elsewhere [1]. Throughout, suppose that G is something that (a) is built out of cells, and (b) can be shelled. To fix ideas, it will be assumed that G is an i-graph on a finite vertex set V. In other words, G is a possibly empty collection of i-element subsets of V. These subsets are the cells, also known as edges.

When a vertex v is removed from V, the cells that contain v must be removed from G. Each of these cells has exactly i elements, one of which is v itself. Thus, the link L_v of G at v is defined to be the $(i-1)$-graph, whose cells are exactly the $(i-1)$-subsets of $V-v$ which, upon the addition of v, become cells of G.

To shell G is to remove the vertices v from V one at a time, until none are left. As this is done, a record is kept of the resulting links L_1, L_2, ..., L_{N-1}, L_N. Here, N is the number of elements in the vertex set V. Each shelling thus determines a sequence $\{L_j\}$ of $(i-1)$-graphs.

The shelling vector $\widetilde{f}G$ is defined inductively as follows. Assume $\widetilde{f}L_i$ is defined, for $(i-1)$-graphs. The sum over all shellings

$$\widetilde{f}G = \sum\nolimits_{\text{shellings}} \widetilde{f}L_1 \otimes \ldots \otimes \widetilde{f}L_N$$

is the shelling vector of G. The induction is founded on 0-graphs. There is only one zero element set, namely the empty set, and it is a subset of any vertex set. Let $a = a_V$ and $b = b_V$ be the two possible 0-graphs on the vertex set V. The 0-graph a has no cells, while b has the empty set as a cell. Use the equations $\widetilde{f}a = a$, $\widetilde{f}b = b$ to found the induction.

The shelling vector is for various reasons too large. The flag vector fG is defined inductively as follows. Much as before, one writes

$$fG = \sum\nolimits_{\text{shellings}} f'L_1 \otimes \ldots \otimes f'L_N$$

where now each $f'L_j$ is a linear function of the flag vector of each link appearing in the shelling. The induction is founded on 0-graphs exactly as before.

The link contributions $f'L$ are defined as follows. Let A and B be two cells of L, that do not have a common vertex. Let $L_{++} = L$, L_{-+}, L_{+-}, L_{--} be the four graphs obtained by removing

*203 Coldhams Lane, Cambridge, CB1 3HY, England. E-mail: j.fine@pmms.cam.ac.uk

none, one or both of A and/or B respectively from L. The flag vector f takes L to a point fL lying in some vector space W. Define W' to be W modulo the span of all expressions of the form

$$fL_{++} - fL_{-+} - fL_{+-} + fL_{--}$$

and then set $f'L$ to be the residue of fL in W'. This completes the definition of i-graph flag vectors.

In the same way, a flag vector can be defined for any finite object, built out of cells, that can be shelled. For ordinary or 2-graphs, the flag vector of graphs on N vertices has $p(N)$, the number of partitions of N, independent components. It satisfies subtle, and as yet unexplored, inequalities. The author hopes that techniques similar to those described in [2] will make these inequalities accessible. Each triangulated n-manifold determines an $(n+1)$-graph. It is hoped that the quantum invariants of a manifold can be expressed as linear functions of the flag vector (of any of its triangulations). All this is discussed further in [1]. The 'disjoint pair of optional cells' rule used to define $f'L$ from fL is similar to the 'independent regional change' concept in Vassiliev theory [3].

References

[1] J. Fine, Quantum topology, hypergraphs and flag vectors, preprint q-alg/9708001 (August 1997)

[2] ——, Convex polytopes and linear algebra, preprint alg-geom/9710001 (October 1997)

[3] ——, Vassiliev theory, regional change, and the Kontsevich integral, (in preparation)